MAGIC RECTANGLES

Marián TRENKLER

On Figure 1 is depicted a spider web, which consists of three nine-gons and nine rays, which cross in 27 points. On those spots are drawn 27 circlets (dew-drops on a spider web).

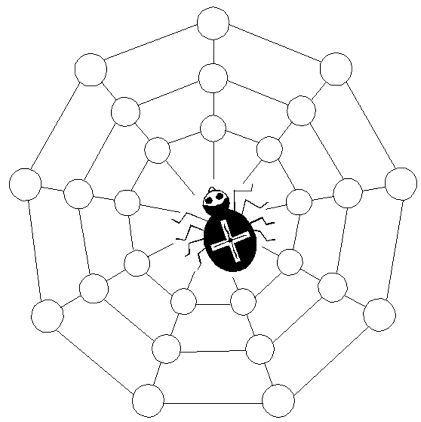

Figure 1.

Problem 1. Inscribe the numbers $1, 2, 3, \ldots, 27$ into the circlets, in such a way, that the sums of numbers on the perimeters of the 9-gons will be the same and also the sums on all the rays will be the same.

Such a problem (a mathematical brain-twister) is usually solved by trial and today we can also use computers. However it would be difficult if we had a web consisting of a hundred 200-gons.

Solution: In Figure 2 is a table consisting of three rows and nine columns. The sums of the numbers in each row and column are the same. If we inscribe the corresponding numbers into the circlets of the web we will get the solution of Problem 1. We explain how the table was made later.

4	18	20	13	27	2	22	9	11
21	5	16	3	14	25	12	23	7
17	19	6	26	1	15	8	10	24

Figure 2.

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Problem 1 can be generalized. Let a spider's web consists of m circles and n rays. We will denote such a web $\mathcal{W}(m,n)$. $\mathcal{T}(m,n)$ will denote a rectangle which consists of $m.n$ squares arranged into m rows and n columns.

Problem 2. Inscribe all the numbers from the set $\{1,2,3,\dots,mn\}$ into the circlets of the web $\mathcal{W}(m,n)$ so that the sums of numbers in the n-gons are the same and the sums on all the rays are the same.

A spider's web which can be evaluated in this way is called a *magic web* and such valuations are called *magic*. We will consider some pairs of parameters m and n for which the problem has or has not a solution in this paper.

A *magic square* $\mathbf{S}_n$ of order n is an $n \times n$ matrix (square table) containing the natural numbers $1,2,\dots,n^2$ in some order, and such that the sum of the number along any row, column, or main diagonal is a fixed constant. In [2] and elsewhere we can find constructions of magic squares of order n for all natural numbers $n \neq 2$. There is not a magic square of order 2, as the reader may easily verify. From the existence of $\mathbf{S}_n$ follows a magic valuation of a web $\mathcal{W}(n,n)$.

In what fallows we will not make use of the diagonal part of thedefenetion of the magic square.

Definition. *A magic rectangle* $\mathbf{M}_{m,n}$ of order m,n is a rectangle $\mathcal{T}(m,n)$ into the squares of which are inscribed all the natural numbers from the set $\{1,2,3,\dots,mn\}$ when the sums in all the rows are the same and the sums in all the columns are the same.

We can suppose without loss of generality that $m \leq n$. It follows directly from the definition that $\mathbf{M}_{1,1}$ exists and $\mathbf{M}_{1,n}$ does not exist for $n \geq 2$.

A magic rectangle $\mathbf{M}_{m,n}$ is made up from mn squares which we denote as $\mathbf{m}(i,j)$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

The sum of all numbers of a magic rectangle $\mathbf{M}_{m,n}$ is

$$\tau = \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbf{m}(i,j) = \frac{1}{2}mn(mn+1).$$

The sum of all numbers in one row of $\mathbf{M}_{m,n}$ is $\rho = \frac{1}{2}n(mn+1)$ and in each column is $\sigma = \frac{1}{2}m(mn+1)$.

Theorem 1. *If one of the numbers m,n is even and the other is odd, then $\mathbf{M}_{m,n}$ does not exists.*

Proof. Without loss of generality we can suppose that m is even and n is odd. The product $n(mn+1)$ is an odd number and therefore σ is not an integer. This is not possible as σ is a sum of integers. $\square$

In individual proofs we describe constructions of corresponding magic rectangles while leaving the verification to the reader.

Theorem 2. *A magic rectangle $\mathbf{M}_{2,2k}$ exists for all $k > 1$.*

Proof. We inscribe numbers $1, 2, 3, \ldots, 2k$ into the first row of table $\mathcal{T}(m, n)$ and numbers $4k, (4k-1), (4k-2), \ldots, (2k+1)$ into the second one. The sums of numbers in all the columns (but not rows) will be the same. Differences of pairs of numbers in individual columns are

$$\{(4k-1), (4k-3), \ldots, 11, 9, 7, 5, 3, 1\}$$

and their sum is $4k^2$. If we exchange the pair of numbers in the j-th column the corresponding difference will change its sign and the sum will decrease. We have to show that we can assign the signs of the differences so that their sum becomes zero.

If k is even (so that the number of columns is a multiple of 4), interchange the pair of numbers in column j if and only if $j = 2(\mod 4)$ or $j = 3(\mod 4)$.

Ik k is odd (≥ 3) proceed as in the even case except that in the last six columns the switch is made in the first and third only. $\square$

Theorem 3. *For all $n > 2$ a magic rectangle $\mathbf{M}_{n,n^2}$ exists.*

Proof. Generate an $n \times n^2$ array as a row, $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots, \mathcal{T}_n$ of n $n \times n$ arrays constructed as follows: row $i+1$ of $\mathcal{T}_j$ is simply the first cyclic shift of row i, and for each $j > 0$, the first row of $\mathcal{T}_{i+1}$ is last row of $\mathcal{T}_j$. This inductive definition of the $n \times n^2$ array is completed by giving the first row of $\mathcal{T}_i$. This is $0, n^2, 2n^2, 3n^2, 4n^2, \ldots, (n-1)n^2$.

Now add (as matrices) a magic square $\mathbf{S}_n$ to each of the $\mathcal{T}_i$. The result is a magic rectangle $\mathbf{M}_{n,n^2}$. $\square$

The construction of $\mathbf{M}_{3,9}$ (on Figure 2) is shown below.

4	9	2
3	5	7
8	1	6

0	9	18	9	18	0	18	0	9
18	0	9	0	9	18	9	18	0
9	18	0	18	0	9	0	9	18

THE CONSTRUCTION OF $\mathbf{M}_{3,9}$

Note. There is another magic rectangle $\mathbf{M}_{n,n^2}$ which we can obtain from a magic cube of order n (see [3]) by cutting it into n layers and inserting into an $n \times n^2$ array.

Corollary. *If a, b are natural numbers with $a.b = n > 2$, then there exists a magic rectangle $\mathbf{M}_{an,bn}$.*

Proof. The case $a = 1$ is just theorem 3.

For $a > 1$ use the same construction as in theorem 3, but arrange the $n \times n$ arrays in the pattern

$$\begin{array}{cccc} \mathcal{T}_1 & \mathcal{T}_2 & \ldots & \mathcal{T}_b \\ \mathcal{T}_{b+1} & \mathcal{T}_{b+2} & \ldots & \mathcal{T}_{2b} \\ & \ldots & \ldots & \\ \mathcal{T}_{(a-1)b+1} & \ldots & \ldots & \mathcal{T}_{ab} \end{array}$$

Theorem 4. *Given magic rectangles $\mathbf{M}_{m,n}$ and $\mathbf{M}_{p,q}$, a magic rectangle $\mathbf{M}_{mp,nq}$ is constructible.*

Proof. Construct the $np \times mq$ array $\mathcal{A}$, partitioned into p rows and q columns of $m \times n$ cells, each of which is $\mathbf{M}_{m,n}$.

Then construct the $mp \times nq$ array $\mathcal{B}$, also partitioned into cells of size $m \times n$. Each cell contains mn identical elements $mn \times [(i,j)$ entry of $\mathbf{M}_{p,q} - 1]$. Then $\mathcal{A} + \mathcal{B}$ is the required magic rectangle. $\square$

It is shown below how a $\mathbf{M}_{6,12}$ is made using a pair of magic rectangles $\mathbf{M}_{2,4}$ and $\mathbf{M}_{3,3}$.

1	7	6	4
8	2	3	5

$\mathbf{M}_{2,4}$

8	1	6
3	5	7
4	9	2

$\mathbf{M}_{3,3}$

57	63	62	60	1	7	6	4	41	47	46	44
64	58	59	61	8	2	3	5	48	42	43	45
17	23	22	20	33	39	38	36	49	55	54	52
24	18	19	21	40	34	35	37	56	50	51	53
25	31	30	28	65	71	70	68	9	15	14	12
32	26	27	29	72	66	67	69	16	10	11	13

$\mathbf{M}_{6,12}$

From the given theorems follows the construction of $\mathbf{M}_{m,n}$ for many pairs of parameters m, n but still there are many pairs of m, n for which we cannot decide whether $\mathbf{M}_{m,n}$ exists. In the following pictures $\mathbf{M}_{3,5}$ and $\mathbf{M}_{3,7}$ are given. You have certainly noticed that the solution of problem 1 is $\mathbf{M}_{3,9}$.

1	10	14	9	6
15	2	7	11	5
8	12	3	4	13

$\mathbf{M}_{3,5}$

1	12	20	8	13	6	17
14	2	10	21	5	16	9
18	19	3	4	15	11	7

$\mathbf{M}_{3,7}$

We conclude with two problem which can be solved by using the previous results.

Problem 3. Prove that $\mathbf{M}_{n,2n}$ exists for all even $n \geq 4$.

Problem 4. Construct $\mathbf{M}_{3,n}$ for some other values of parameter $n \geq 11$.

Literature

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P.J.Šafárik University, Jesenná 5, 041 54 Košice, Slovakia
E-mail address: trenkler@duro.science.upjs.sk